Homework 4 Solution

1. (a) Find two linear transformations $T : V \longrightarrow W$ and $U : W \longrightarrow V$ such that $UT = T_0$ (the zero transformation from V to V) but $TU \neq T_0$ (the zero transformation from W to W).

   (b) Based on T and U in (a), find two matrices A and B such that $AB = O$ but $BA \neq O$.

Solution.

(a) Let $V = \mathbb{R}^2$ $W = \mathbb{R}^2$

$T : V \to W$ $U : W \to V$

$(x, y) \mapsto (0, x)$ $(x, y) \mapsto (x, 0)$

Then $UT(x, y) = U(0, x) = (0, 0)$ $\forall (x, y) \in V$

So $UT = T_0$

$TU(x, y) = T(x, 0) = (0, x)$ $\forall (x, y) \in W$

So $TU \neq T_0$

(b)

Let $\beta = \{e_1, e_2\}$ be the standard ordered basis for $\mathbb{R}^2$

$A = [U]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $B = [T]_\beta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Then $AB = O_{2\times 2}$

$BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq O_{2\times 2}$

2. Let $g_0(x) = x + 1$. Let $T : P_2(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ and $U : P_3(\mathbb{R}) \longrightarrow \mathbb{R}^3$ be defined by

$$T(f(x)) = f'(x)g_0(x) + \int_0^x f(t)dt \text{ and } U(h(x)) = (h(0), h(1), h'(1))$$

Let α, β, γ be the standard ordered bases for $P_2(\mathbb{R}), P_3(\mathbb{R}), \mathbb{R}^3$ respectively.

(a) Compute $[T]_\alpha^\beta$, $[U]_\beta^\gamma$, $[U]_\beta^\gamma[T]_\alpha^\beta$ and $[UT]_\alpha^\gamma$.

(b) Let $h_0(x) = 1 - 2x - x^2 + x^3$, compute $[h_0(x)]_\beta$, $[U]_\beta^\gamma[h_0(x)]_\beta$ and $[U(h_0(x))]_\gamma$.

Solution.

$\alpha = \{1, x, x^2\}$ $\quad \beta = \{1, x, x^2, x^3\}$ $\quad \gamma = \{e_1\ e_2\ e_3\}$

(a) $[T]_\alpha^\beta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 1/2 & 2 \\ 0 & 0 & 1/3 \end{pmatrix}$ $\quad [U]_\beta^\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$

$$[U]_\beta^\gamma \cdot [T]_\alpha^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 1/2 & 2 \\ 0 & 0 & 1/3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 5/2 & 13/3 \\ 1 & 2 & 7 \end{pmatrix}$$

$$[UT]_\alpha^\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 5/2 & 13/3 \\ 1 & 2 & 7 \end{pmatrix}$$

(b) $[h_0(x)]_\beta = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 1 \end{pmatrix}$

$$[U]_\beta^\gamma [h_0(x)]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$[U(h_0(x))]_\gamma = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

3. Sec. 2.3: Q17

17. Let V be a vector space. Determine all linear transformations $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ such that $\mathsf{T} = \mathsf{T}^2$. *Hint:* Note that $x = \mathsf{T}(x) + (x - \mathsf{T}(x))$ for every x in V, and show that $\mathsf{V} = \{y\colon \mathsf{T}(y) = y\} \oplus \mathsf{N}(\mathsf{T})$ (see the exercises of Section 1.3).

Proof. Claim: $T = T^2 \iff T$ is a projection

($\Rightarrow$) suppose $T = T^2$. Let $W_T = \{y \in V : T(y) = y\}$

We will show that $V = W_T \oplus N(T)$

① $\forall x \in W_T \cap N(T)$.

$$\begin{cases} x \in W_T, & \text{so } x = T(x). \\ x \in N_T, & \text{so } T(x) = 0 \end{cases} \Rightarrow x = 0$$

so $W_T \cap N(T) = \{0\}$

② $\forall x \in W_T$, $x = T(x) \in R(T)$, so $W_T \subset R(T)$

$\forall x \in V$, $T(T(x)) = T^2(x) = T(x)$ so $T(x) \in W_T$ i.e. $R(T) \subset W_T$

Therefore $W_T = R(T)$

Besides, $T(x - T(x)) = T(x) - T^2(x) = 0 \quad \forall x \in V$

so $x - T(x) \in N(T) \quad \forall x \in V$.

Since $x = \underbrace{T(x)}_{\in W_T} + \underbrace{(x - T(x))}_{\in N(T)} \quad \forall x \in V$

then $V = W_T + N(T)$

By ① and ②, $V = W_T \oplus N(T) = R(T) \oplus N(T)$

T is projection on W_T along $N(T)$

($\Leftarrow$) suppose T is a projection on W_1 along W_2.

then $V = W_1 \oplus W_2$

For any $v \in V$. $\exists !\ w_1 \in W_1$, $w_2 \in W_2$

s.t. $v = w_1 + w_2$, and $T(v) = w_1$

Since $\underset{\substack{\Uparrow \\ W_1}}{w_1} = w_1 + \underset{\substack{\Uparrow \\ W_2}}{0}$, we have $T(w_1) = w_1$

So $T^2(v) = T(T(v)) = T(w_1) = w_1 = T(v)$

i.e $T^2 = T$

4. Let V and W be two finite-dimensional vector spaces, and let $T : V \longrightarrow W$ be a linear transformation. Suppose β is a basis for V. Prove that T is invertible if and only if $T(\beta)$ is a basis for W.

Proof:

($\Rightarrow$) Suppose T is invertible (injective and surjective)

then $\dim(V) = \dim(W) = n < \infty$

Since β is L.I. and T is injective.

we have $T(\beta)$ is L.I.

Since T is surjective. $W = R(T) = \operatorname{span}(T(\beta))$

Thus. $T(\beta)$ is a basis for W.

($\Leftarrow$) Suppose $T(\beta)$ is a basis for W.

$W = \operatorname{span}(T(\beta)) = R(T)$, so T is surjective

$\dim(V) = |\beta| = |T(\beta)| = \dim(W)$

So. $\dim(N(T)) = \dim(V) - \dim(R(T))$

$= \dim(V) - \dim(W)$

$= 0$

i.e $N(T) = \{0\}$, T is injective

Thus T is invertible.

5. Sec. 2.4: Q16

16. Let B be an $n \times n$ invertible matrix. Define $\Phi\colon \mathsf{M}_{n\times n}(F) \to \mathsf{M}_{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

Proof:

- Φ is linear.

  $\forall A_1, A_2 \in M_{n\times n}(F)$. $\forall a \in F$.

  $$\begin{aligned}\Phi(a\cdot A_1 + A_2) &= B^{-1}(aA_1 + A_2)B \\ &= B^{-1}\cdot(aA_1B + A_2B) \\ &= a\cdot B^{-1}A_1B + B^{-1}A_2B \\ &= a\cdot \Phi(A_1) + \Phi(A_2)\end{aligned}$$

- Φ is injective

  $\forall A \in N(\Phi)$. $\Phi(A) = O_{n\times n}$ i.e. $B^{-1}AB = O_{n\times n}$

  Since B is invertible. $B\cdot B^{-1} = B^{-1}B = I$

  $A = B\cdot(B^{-1}AB)\cdot B^{-1} = B\cdot O_{n\times n}\cdot B^{-1} = O_{n\times n}$

  Thus $N(\Phi) = \{O_{n\times n}\}$

- Φ is surjective

  $\forall A \in M_{n\times n}(F)$. $\exists\ BAB^{-1} \in M_{n\times n}(F)$

  s.t $\Phi(BAB^{-1}) = B^{-1}\cdot(BAB^{-1})\cdot B = A$

  Thus $R(\Phi) = M_{n\times n}(F)$.

In all, Φ is an isomorphism.